Chapter 1. Electrostatics I

Notes:

- *Most of the material presented in this chapter is taken from Jackson, Chap. 1.*
- *Units from the Système International (SI) will be used in this chapter.*

1.1 Mathematical Considerations

1.1.1 Dirac's delta function (distribution)

a) Dirac's delta function is defined such that

$$\delta(x-a)=\begin{cases}\infty, & \text{for } x=a\\ 0, & \text{for } x\neq a\end{cases} \tag{1.1}$$

but with the restriction that the "area" it encloses equals 1. Therefore, whenever it is used it is implied that

$$\int_{-\infty}^{\infty}\delta(x-a)dx=1. \tag{1.2}$$

b) It is also required from the delta function that for any good function $g(x)$ (i.e., one that is differentiable everywhere any number of times, while it and its derivatives vanish faster than any power of $1/|x|$ as $|x|\to\infty$) we have

$$\int_{-\infty}^{\infty}g(x)\delta(x-a)dx=g(a). \tag{1.3}$$

Dirac's function can be approximated with many different functions. For example, the function

$$D_n(x)=\sqrt{\frac{n}{\pi}}\,e^{-nx^2} \qquad n=1,2,\ldots \tag{1.4}$$

can be shown to satisfy equation (1.3) when $n\to\infty$. To prove this, we first set $y=x-a$ in the integrand of equation (1.3), and we replace $\delta(y)$ by $D_n(y)$

$$\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}g(y+a)e^{-ny^2}dy=g(a). \tag{1.5}$$

We next evaluate the following (with $|\cdots|$ denoting the absolute value)

$$\left|\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} g(y+a)e^{-ny^2}dy - g(a)\right| = \sqrt{\frac{n}{\pi}}\left|\int_{-\infty}^{\infty} e^{-ny^2}\left[g(y+a)-g(a)\right]dy\right|, \tag{1.6}$$

since

$$\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-ny^2}dy = 1. \tag{1.7}$$

Because it is always the case that

$$\left|g(y+a)-g(a)\right| \le |y|\cdot\max\left\{\left|\frac{dg}{dy}\right|\right\}, \tag{1.8}$$

then,

$$\begin{aligned}\sqrt{\frac{n}{\pi}}\left|\int_{-\infty}^{\infty} e^{-ny^2}\left[g(y+a)-g(a)\right]dy\right| &\le \max\left\{\left|\frac{dg}{dy}\right|\right\}\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty}|y|e^{-ny^2}dy \\ &\le \frac{1}{\sqrt{n\pi}}\max\left\{\left|\frac{dg}{dy}\right|\right\} \underset{n\to\infty}{\to} 0.\end{aligned} \tag{1.9}$$

Equation (1.3) is, therefore, verified with any good function $g(x)$ for $D_\infty(x)$.

c) The derivative of the delta function can also be defined and used as follows

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)\delta'(x-a)dx &= g(x)\delta(x-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g'(x)\delta(x-a)dx \\ &= -g'(a),\end{aligned} \tag{1.10}$$

where we integrated by parts and $g' = dg/dx$.

d) Since from equation (1.1) $\delta(x)$ is zero everywhere except at $x = 0$, we can write

$$\int_{-\infty}^{\infty}\delta(x-a)dx = \int_{a-\varepsilon}^{a+\varepsilon}\delta(x-a)dx, \tag{1.11}$$

with $\varepsilon > 0$. Then for a function $y(x)$ with multiple zeros

$$\int_{-\infty}^{\infty} g(x)\delta\left[y(x)\right]dx = \sum_i \int_{y(x_i)-\varepsilon}^{y(x_i)+\varepsilon} g\left[x(y)\right]\delta(y)\frac{dy}{\left|\frac{dy}{dx}\right|}, \tag{1.12}$$

since $dx = \frac{dx}{dy}dy$. We can, therefore, write

$$\delta\left[y(x)\right]=\sum_i \frac{\delta(x-x_i)}{\left|\frac{dy}{dx}\right|_{x=x_i}}, \tag{1.13}$$

where the x_i's are the roots of the equation $y(x)=0$.

e) Dirac's distribution is the derivative of the so-called **Heaviside step function** $H(x)$ (also a distribution)

$$H(x)=\begin{cases}1, & \text{for } x>0\\ 0, & \text{for } x<0\end{cases} \tag{1.14}$$

For a good function $g(x)$

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dH(x)}{dx}g(x)dx &= H(x)g(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}H(x)\frac{dg(x)}{dx}dx\\ &= -\int_0^{\infty}\frac{dg(x)}{dx}dx=g(0).\end{aligned} \tag{1.15}$$

From equations (1.3) and (1.15) we see that

$$\delta(x)=\frac{dH(x)}{dx}. \tag{1.16}$$

f) Finally, higher-dimensional versions of the delta function are the product of delta functions from each dimension. For example, in three-dimension

$$\delta(\mathbf{x}-\mathbf{X})=\delta(x_1-X_1)\delta(x_2-X_2)\delta(x_3-X_3) \tag{1.17}$$

vanishes everywhere except at $\mathbf{x}=\mathbf{X}$, and

$$\int_{\Delta V}\delta(\mathbf{x}-\mathbf{X})d^3x=\begin{cases}1, & \text{if } \Delta V \text{ contains } \mathbf{x}=\mathbf{X}\\ 0, & \text{if } \Delta V \text{ does not contain } \mathbf{x}=\mathbf{X}\end{cases} \tag{1.18}$$

1.1.2 Tensor notation

The **Kronecker tensor** is defined by

$$\delta_{jk} = \begin{cases} 1, & \text{for } j = k \\ 0, & \text{for } j \neq k \end{cases} \tag{1.19}$$

where j and k are tensor indices (e.g., in a three-dimensional space the indices can take the value 1, 2, or 3). This tensor is useful to compactly express some vector expressions. For example, the dot product of two vectors can be written as

$$\mathbf{A} \cdot \mathbf{B} = \delta_{jk} A_k B_j = A_j B_j, \tag{1.20}$$

where a summation on repeated indices is implied (i.e., $A_j B_j \equiv \sum_j A_j B_j$).

The **Levi-Civita tensor** is defined by

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{even permutation of } ijk \\ -1, & \text{odd permutation of } ijk \\ 0, & \text{if } i = j,\, i = k, \text{ or } j = k \end{cases} \tag{1.21}$$

For example, $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$, $\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$, and $\varepsilon_{113} = \varepsilon_{121} = \varepsilon_{133} = \cdots = 0$. This tensor can be used to compactly express the cross product

$$\left[\mathbf{A} \times \mathbf{B}\right]_i = \varepsilon_{ijk} A_j B_k. \tag{1.22}$$

There also exist many relations that link the Kronecker and Levi-Civita tensors. Most notably,

$$\varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}. \tag{1.23}$$

As examples of the usefulness of these tensors, consider the following vector formulas

$$\begin{aligned} \left[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})\right]_i &= \varepsilon_{ijk} A_j \varepsilon_{kmn} B_m C_n \\ &= \varepsilon_{kij} \varepsilon_{kmn} A_j B_m C_n \\ &= \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) A_j B_m C_n \\ &= A_j C_j B_i - A_j B_j C_i \\ &= (\mathbf{A} \cdot \mathbf{C}) B_i - (\mathbf{A} \cdot \mathbf{B}) C_i, \end{aligned} \tag{1.24}$$

or,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}, \tag{1.25}$$

and with ϕ and ψ two scalar functions

$$\begin{aligned}\nabla\cdot(\phi\nabla\psi)&=\partial_i(\phi\partial_i\psi)\\&=(\partial_i\phi)(\partial_i\psi)+\phi\partial_i\partial_i\psi\\&=(\nabla\phi)\cdot(\nabla\psi)+\phi\nabla^2\psi,\end{aligned} \tag{1.26}$$

where $\partial_i\equiv\dfrac{\partial}{\partial x_i}$.

1.1.3 The divergence theorem

Consider an infinitesimal volume $\Delta^3 x$ bounded by the surface of a cube whose sides have an area Δa (see Figure 1.1). Defining $\mathbf{n}_i$ as a unit vector pointing out of the volume element (with $i=1,2,3$ or x,y,z) and therefore normal to the two (parallel) surfaces that are perpendicular to the x_i-axis, we are interested in calculating the following quantity

$$\sum_{\substack{\text{over six}\\\text{surfaces}}}\mathbf{D}\cdot\mathbf{n}_i\,\Delta a, \tag{1.27}$$

where $\mathbf{D}$ is some arbitrary vector, and the summation is taken over the six sides of the cube. Concentrating first on the surfaces corresponding to $i=1$ or x, and setting the value of the corresponding component of $\mathbf{D}$ at the "back" surface of Figure 1.1 to D_x, we can approximate that at the "front" surface (located a distance Δx ahead of the previous one) we will have $D_x+\partial D_x/\partial x\,\Delta x$ for the same component of $\mathbf{D}$. We can then write the corresponding contribution to equation (1.27) as

$$\left[\left(D_x+\frac{\partial D_x}{\partial x}\Delta x\right)-D_x\right]\Delta y\Delta z=\frac{\partial D_x}{\partial x}\Delta x\Delta y\Delta z=\frac{\partial D_x}{\partial x}\Delta^3 x, \tag{1.28}$$

since the unit normal vector to the front (back) surface is along $\mathbf{e}_x$ ($-\mathbf{e}_x$). Using a similar procedure for the other four surfaces, it can easily be shown that

$$\sum_{\substack{\text{over six}\\\text{surfaces}}}\mathbf{D}\cdot\mathbf{n}_i\,\Delta a=(\nabla\cdot\mathbf{D})\Delta^3 x. \tag{1.29}$$

We can extend this treatment to an arbitrary large volume V provided that we divide it into an ensemble of adjacent infinitesimal cubes. If we further denote by S the closed surface that contains V, we can finally write the so-called **divergence theorem** (sometimes called **Gauss' theorem**) as

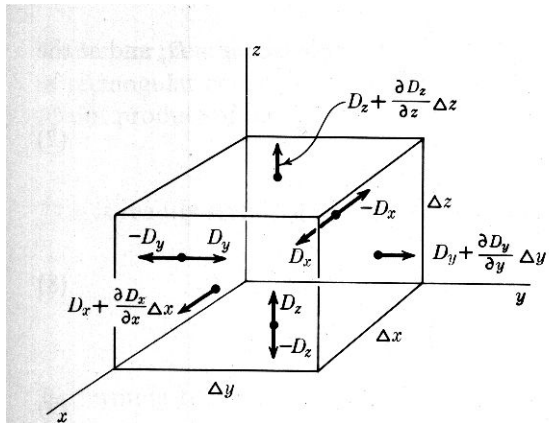

Figure 1.1 – Illustration of the divergence theorem.

$$\boxed{\int_V \nabla \cdot \mathbf{D}\, d^3x = \int_S \mathbf{D} \cdot \mathbf{n}\, da} \tag{1.30}$$

The generalization from equation (1.29) to equation (1.30) arises from the fact that the sides of every infinitesimal cube that is *interior* to the volume V will cancel with the side of an adjacent cube, leaving only the *exterior* side of the cubes that make out the closed surface S.

Similarly, if we consider the components of $\mathbf{D}$ which are parallel to the surfaces of the infinitesimal cubes (i.e., $\mathbf{n}_i \times \mathbf{D}$), it can similarly be shown that

$$\boxed{\int_V \nabla \times \mathbf{D}\, d^3x = \int_S \mathbf{n} \times \mathbf{D}\, da} \tag{1.31}$$

Finally, we could apply the same procedure to a scalar function ψ to get

$$\boxed{\int_V \nabla \psi\, d^3x = \int_S \psi \mathbf{n}\, da} \tag{1.32}$$

1.1.4 **Stokes' theorem**

Let's consider an open surface S with an exterior right-handed contour C that we divide into a large number of infinitesimal square loops (see Figure 1.2). We are interested in evaluating the following quantity

$$\sum_{\substack{\text{over four}\\ \text{segments}}} \mathbf{E} \cdot d\mathbf{l}, \tag{1.33}$$

where $\mathbf{E}$ is some vector, and $\mathbf{l}$ the right-handed contour of the infinitesimal loop (made up with four segment vectors) over which the summation is taken. If we denote the

"bottom" (top) segment vector by $\mathbf{a}$ ($-\mathbf{a}$) and the "right" (left) one by $\mathbf{b}$ ($-\mathbf{b}$), we can approximate equation (1.33) with

$$\begin{aligned}
\sum_{\substack{\text{over four}\\ \text{segments}}} \mathbf{E}\cdot d\mathbf{l} &= \underset{\text{bottom}}{\mathbf{E}\cdot\mathbf{a}} - \underset{\text{top}}{\left[\mathbf{E}+(\mathbf{b}\cdot\nabla)\mathbf{E}\right]\cdot\mathbf{a}} + \underset{\text{right}}{\left[\mathbf{E}+(\mathbf{a}\cdot\nabla)\mathbf{E}\right]\cdot\mathbf{b}} - \underset{\text{left}}{\mathbf{E}\cdot\mathbf{b}} \\
&= \mathbf{b}\cdot\left[(\mathbf{a}\cdot\nabla)\mathbf{E}\right] - \mathbf{a}\cdot\left[(\mathbf{b}\cdot\nabla)\mathbf{E}\right] \\
&= b_k a_j \partial_j E_k - a_j b_k \partial_k E_j \\
&= a_j b_k \left(\partial_j E_k - \partial_k E_j\right) \\
&= a_j b_k \partial_m E_n \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) \\
&= \left(\varepsilon_{ijk} a_j b_k\right)\left(\varepsilon_{imn}\partial_m E_n\right) \\
&= (\mathbf{a}\times\mathbf{b})\cdot(\nabla\times\mathbf{E}),
\end{aligned} \tag{1.34}$$

where we have resorted to the tensor notation and used equation (1.23). If we further define $\mathbf{n}$ as the unit vector normal to the loop's surface and Δa as the loop's area, we have

$$\mathbf{a}\times\mathbf{b} = \mathbf{n}\,\Delta a. \tag{1.35}$$

Inserting equation (1.35) into equation (1.34), and combining it with the results obtained for all the other loops (included in S) using this same procedure, we can write the so-called **Stokes' theorem**

$$\boxed{\int_S (\nabla\times\mathbf{E})\cdot\mathbf{n}\,da = \oint_C \mathbf{E}\cdot d\mathbf{l}} \tag{1.36}$$

The generalization of the results obtained for one loop to the whole surface S is possible because of the fact that contributions along common segments of adjacent infinitesimal loops that are *interior* to the surface S will cancel each other, leaving only contributions along the *exterior* sides of the loops that make out the closed circuit C.

Finally, applying the same technique to the case of a scalar function ψ will yield the following relation

$$\boxed{\int_S \mathbf{n}\times\nabla\psi\,da = \oint_C \psi\,d\mathbf{l}} \tag{1.37}$$

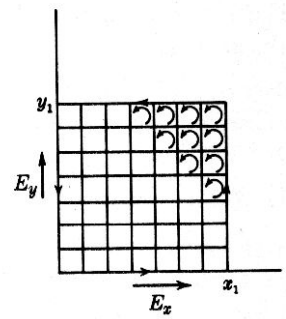

Figure 1.2 – Illustration of Stokes' theorem.

1.1.5 **Green's theorem**

We start with equation (1.30) for the divergence theorem

$$\int_V \nabla \cdot \mathbf{D}\, d^3x = \int_S \mathbf{D} \cdot \mathbf{n}\, da, \tag{1.38}$$

and we let $\mathbf{D} = \phi \nabla \psi$, where ϕ and ψ are scalar functions. Using equation (1.26) we have

$$\nabla \cdot (\phi \nabla \psi) = (\nabla \phi) \cdot (\nabla \psi) + \phi \nabla^2 \psi \tag{1.39}$$

for the integrand of the left-hand side of equation (1.38), and for the integrand of the right-hand side

$$\phi \nabla \psi \cdot \mathbf{n} = \phi \left[(\mathbf{n} \cdot \nabla) \psi \right] = \phi \frac{\partial \psi}{\partial n}, \tag{1.40}$$

where $\partial / \partial n$ is the outward derivative in a direction normal to the surface S. Inserting equations (1.39) and (1.40) into equation (1.38) yields **Green's first identity**

$$\boxed{\int_V \left[\phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi) \right] d^3x = \int_S \phi \frac{\partial \psi}{\partial n}\, da} \tag{1.41}$$

Interchanging ϕ and ψ in equation (1.41) and subtracting the result from the original equation, we get **Green's theorem**

$$\boxed{\int_V \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi \right] d^3x = \int_S \left[\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] da} \tag{1.42}$$

1.2 Coulomb's Law

Coulomb's Law states that *the force acting between two charged bodies at rest with respect with each other is inversely proportional to the square of the distance between them, proportional to each charge, directed along the line joining them, and repulsive for like charges and attractive for opposite charges*.

Mathematically speaking the force defined in Coulomb's Law is expressed as follows

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}, \tag{1.43}$$

where $\mathbf{x}_i$ is the position of the charge q_i, and $(4\pi\varepsilon_0)^{-1} = 10^{-7} c^2$ so that the permittivity of vacuum $\varepsilon_0 \simeq 8.854 \times 10^{-12}$ (F/m) (c is the speed of light in vacuum, and F stands for "Farad"). The SI unit for charge is the *coulomb* (C); the electron's charge is $q \simeq 1.602 \times 10^{-19}$ C.

The **electric field** is defined as the force per unit charge at a given point. That is,

$$\mathbf{F} = q\mathbf{E}, \tag{1.44}$$

with $\mathbf{E}$ the electric field (which has units of Volts per meter (V/m)). Combining equation (1.43) and (1.44), it is seen that a charge q_1 generates an electric field, which according to Coulomb's Law is given by

$$\mathbf{E}(\mathbf{x}) = \frac{q_1}{4\pi\varepsilon_0} \frac{\mathbf{x} - \mathbf{x}_1}{|\mathbf{x} - \mathbf{x}_1|^3}. \tag{1.45}$$

When an aggregate of charge charges is involved, the field generated is simply calculated from the linear superposition of the field generated by each charge

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \sum_i q_i \frac{\mathbf{x} - \mathbf{x}_1}{|\mathbf{x} - \mathbf{x}_1|^3}. \tag{1.46}$$

Alternatively, in the continuous limit where the charge density $\rho(\mathbf{x})$ (i.e., the amount of charge per unit volume) is given, then

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \, d^3x'. \tag{1.47}$$

1.3 Gauss' Law and the Electric Field

We consider the electric field generated by a single point charge q that is enclosed within an arbitrary surface S. Let's denote by da an infinitesimal surface element at a

given point on the surface, and located a distance r away from the charge. If the electric field at that point makes an angle θ with the normal vector $\mathbf{n}$ to da, then we can write from equation (1.45)

$$\begin{aligned}\mathbf{E}\cdot\mathbf{n}\,da &= \frac{q}{4\pi\varepsilon_0}\frac{\cos(\theta)}{r^2}\,da \\ &= \frac{q}{4\pi\varepsilon_0}d\Omega,\end{aligned} \tag{1.48}$$

since the solid angle subtended by da at the charge is given by $r^2 d\Omega = \cos(\theta)da$. If we now integrate over the whole surface we get

$$\int_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{\varepsilon_0}. \tag{1.49}$$

It is important to realize that *if the charge were located outside of the volume V, then the right-hand side of equation (1.49) would be zero*. Equation (1.49) can be generalized to cases involving multiple charges

$$\int_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\sum_i q_i, \tag{1.50}$$

or a charge density

$$\boxed{\int_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x}')\,d^3x'} \tag{1.51}$$

where V is the volume enclosed by S. The right-hand side of equation (1.51) represents the total charge contained in V. Equation (1.51) is **Gauss' Law** for a charge density.

We can transform this last equation using the divergence theorem (i.e., equation (1.30)) to get

$$\int_V \left(\nabla\cdot\mathbf{E} - \frac{\rho}{\varepsilon_0}\right) d^3x' = 0. \tag{1.52}$$

But since the volume is arbitrary, equation (1.52) implies that

$$\boxed{\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}} \tag{1.53}$$

which is the differential form of Gauss' Law of electrostatics.

1.4 The Scalar Potential Function

We now go back to equation (1.47) for the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x', \tag{1.54}$$

and we note that part of the integrand can be simplified as follows

$$\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right). \tag{1.55}$$

Inserting equation (1.55) into equation (1.54) we find a new equation for the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{-1}{4\pi\varepsilon_0}\nabla\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x'. \tag{1.56}$$

We are now in a position to introduce the **scalar potential** function $\Phi(\mathbf{x})$ (with units of Volts (V))

$$\boxed{\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x'} \tag{1.57}$$

from which the electric field can be calculated with

$$\boxed{\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})} \tag{1.58}$$

Since it is always true that for any scalar function $\Phi(\mathbf{x})$ we have $\nabla\times\nabla\Phi = 0$, then it follows from equation (1.58) that

$$\boxed{\nabla\times\mathbf{E} = 0} \tag{1.59}$$

If we consider the work that is required to move a test charge q from one point to another in a given electric field, we have

$$\begin{aligned} W &= -\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F}\cdot d\mathbf{l} = -q\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{E}\cdot d\mathbf{l} \\ &= q\int_{\mathbf{x}_1}^{\mathbf{x}_2} \nabla\Phi\cdot d\mathbf{l} = q\int_{\mathbf{x}_1}^{\mathbf{x}_2} d\Phi \\ &= q\left[\Phi(\mathbf{x}_2) - \Phi(\mathbf{x}_1)\right]. \end{aligned} \tag{1.60}$$

We see from this last result that we can equate the quantity $q\Phi$ to the potential energy of the test charge.

Example

Use Gauss' Law and equation (1.59) to prove the following:

a) Any excess charge placed on a conductor must lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields.)

b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shields its exterior from the fields due to charges placed inside it.

c) The electric field at the surface of a conductor is normal to the surface and has a magnitude of σ/ε_0, where σ is the charge density per unit area on the surface (i.e., it is the surface charge density).

Solution.

a) Because of the given definition, the electric field inside a conductor must always be zero, since the sudden movement of the free moving charges within the conductor will quickly counterbalance any applied electric field. Now, let's define a closed surface S that encloses the conductor completely, as well as some small amount of volume exterior to the conductor. From Gauss' Law (i.e., equation (1.49)) we can write

$$\int_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{\varepsilon_0}, \tag{1.61}$$

where q is the excess charge placed on the conductor. We further break the volume enclosed by S into two sub-volumes. The first one is delimited by the closed surface S_1, which is entirely contained within the conductor but does not include its boundary, while the other, delimited by a surface S_2, enclosing the boundary and the remainder of the volume within S that is exterior to the conductor. We can therefore write

$$\int_S \mathbf{E}\cdot\mathbf{n}\,da = \int_{S_1} \mathbf{E}\cdot\mathbf{n}_1\,da_1 + \int_{S_2} \mathbf{E}\cdot\mathbf{n}_2\,da_2 = \frac{q}{\varepsilon_0}, \tag{1.62}$$

but since the electric field in the conductor is known to be zero, then equation (1.62) simplifies to

$$\int_{S_2} \mathbf{E}\cdot\mathbf{n}_2\,da_2 = \frac{q}{\varepsilon_0}. \tag{1.63}$$

It is now straightforward to see that through a limiting process we could modify the boundary of S_2 (and by the same token S_1) such that the volume it encloses is limited to the surface of the conductor. It follows that the excess charge has to be located on this surface.

b) We consider three volumes: one enclosing the cavity at the interior of the conductor (with a boundary S_1 that does not include the inner boundary of the conductor), a second (of boundary S_2) encloses entirely, and only, the conductor, and another (boundary S_3) enclosing the volume exterior to the conductor. If we first place a charge q_{int} within S_1 the free moving charges of the conductor will rearrange themselves such that the field in the conductor is zero. That is, charges of polarity opposite to q_{int} will migrate to the inner surface of the conductor in such a way that the electric field in the conductor is zero everywhere. Furthermore, because of Gauss' Law the total charge on the inner surface must equal $-q_{\text{int}}$. But since the conductor must remain globally neutral, it must be that a charge of $+q_{\text{int}}$ will migrate to the outer surface of the conductor. Then, by Gauss's Law the electric field in S_3 caused by the charges within S_1 and S_2 must obey the following relation

$$\int_{S_1} \mathbf{E} \cdot \mathbf{n}_1 \, da_1 + \int_{S_2} \mathbf{E} \cdot \mathbf{n}_2 \, da_2 = \frac{q_{\text{int}}}{\varepsilon_0}. \tag{1.64}$$

That is, the region external to the conductor is not shielded by charges located within the cavity.

On the other hand, if we place a charge q_{ext} in the volume contained in S_3, then free moving charges within the conductor will migrate to the outer surface of the conductor to ensure that the electric field is zero everywhere within the conductor. More precisely, charges of polarity opposite to q_{ext} will be located on a part of the outer surface of the conductor that is closer to the external charge (i.e., where $\cos(\theta) < 0$ in equation (1.48)), and charge of like polarity to q_{ext} will be on the rest of the outer surface (i.e., where $\cos(\theta) > 0$ in equation (1.48)). The important thing to realize is that the total charge on the outer surface of the conductor is equal to zero, and since the electric field within the conductor is also zero, then there will be no migration of charges to the inner surface of the conductor (caused by the presence of q_{ext} outside the conductor). That is, the volume located within the outer boundary of the conductor is unaffected by q_{ext} owing to the rearrangement of charges on that boundary, and the cavity is shielded from charges external to the conductor.

c) We first refer to the infinitesimal pillbox of Figure 1.3 that straddles the outer boundary of the conductor with S. From Gauss' Law we know that

$$\int_S \mathbf{E} \cdot \mathbf{n} \, da = \frac{q}{\varepsilon_0}, \tag{1.65}$$

where S is the closed surface delimiting the pillbox, $\mathbf{n}$ is the unit vector normal to S, and q is the total charge located inside the box. In the limit where we let the sides of the pillbox shrink to an infinitesimally small size, q is then completely located on the boundary, and we can simplify equation (1.65) to

$$\left(\mathbf{E}_{\perp} - \mathbf{E}_{\text{cond}}\right) \cdot \mathbf{n}\, \Delta a = \frac{q}{\varepsilon_0}, \tag{1.66}$$

with Δa the area of the top and bottom surface of the pillbox. We know, however, that the electric field $\mathbf{E}_{\text{cond}}$ inside the conductor is zero. We then find that the magnitude of the electric field $\mathbf{E}_{\perp}$ perpendicular to the surface of the connector is

$$E_{\perp} = \frac{\sigma}{\varepsilon_0}, \tag{1.67}$$

where $\sigma = q/\Delta a$ is the surface charge density. Finally, considering the infinitesimal loop of Figure 1.3, and letting their sides shrink to an infinitesimally small size, we find that the tangential electric field at the surface of the conductor is zero. We can ascertain this by integrating the electric field around the loop, and using Stokes' theorem with equation (1.59). More precisely,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_A \nabla \times \mathbf{E}\, da = 0, \tag{1.68}$$

where A is the (open) surface covered by the loop. When the sides of the loop are infinitesimally small we have (remembering that electric field inside the conductor is zero)

$$\mathbf{E}_{\parallel} \cdot \Delta \mathbf{l} = 0. \tag{1.69}$$

The combination of equations (1.67) and (1.69) the fact the electric field at the surface of the conductor is perpendicular to the surface and has a magnitude of σ/ε_0.

1.5 The Equations of Laplace and Poisson

If we start with equation (1.53) for the divergence of the electric field (i.e., $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$) and we combine it with equation (1.58) relating $\mathbf{E}$ to the potential (i.e., $\mathbf{E} = -\nabla\Phi$), we get the so-called **Poisson equation**

$$\boxed{\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}} \tag{1.70}$$

In regions where charges are absent, we have the **Laplace equation**

$$\boxed{\nabla^2\Phi = 0} \tag{1.71}$$

Incidentally, the Poisson equation can be used to evaluate an important relation that connects the function $1/r$, which often appears in electrostatics, and the Dirac delta function. If we consider the scalar potential at a position $\mathbf{x}$ due to a single point charge q located at $\mathbf{x}''$, from equation (1.70) and $\rho(\mathbf{x}) = q\,\delta(\mathbf{x}-\mathbf{x}'')$ we get

$$\nabla^2\Phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{x}-\mathbf{x}''). \tag{1.72}$$

But we also know from equation (1.57) that

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}''|}. \tag{1.73}$$

Hence, it must be the case that

$$\boxed{\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}''|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}'')} \tag{1.74}$$

1.6 Boundary Conditions, and Single- and Double-layer Surface-charge Distributions

Let's consider the interface between two different media delimited by a surface S (with a unit normal vector $\mathbf{n}$ directed from medium 1 to medium 2) on which a surface-charge density $\sigma(\mathbf{x})$ is distributed. Let the volume V enclosed by the surface be that of a small pillbox (see Figure 1.3). Using equation (1.51) (i.e., Gauss' Law) we can write

$$\int_S \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{x}')\,d^3x'. \tag{1.75}$$

In the limit where the height of the pillbox is made infinitesimally small, so that its end surfaces (of area Δa) are just on either side of the boundary, we have

$$(\mathbf{E}_2 - \mathbf{E}_1)\cdot\mathbf{n}\,\Delta a = \frac{\sigma}{\varepsilon_0}\,\Delta a. \tag{1.76}$$

We therefore find that *there is a discontinuity in the normal component of the electric at the boundary between the two media, which is proportional to the surface-charge density*.

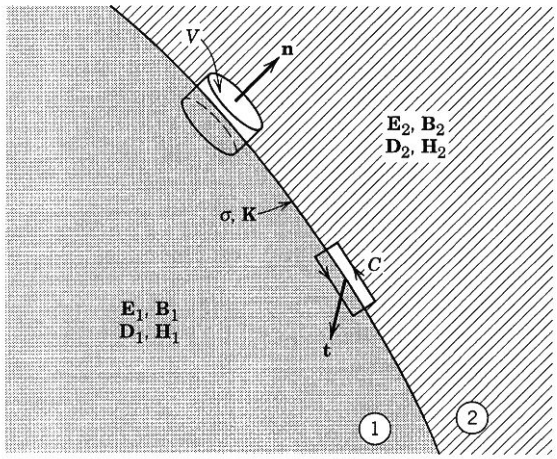

Figure 1.3 – The boundary surface between two media. We determine the boundary conditions for the electric field and the scalar potential by considering a small pillbox S and a small rectangular contour C.

Alternatively, we consider a rectangular loop C that straddles the boundary as shown in Figure 1.3 (with the vector $\mathbf{t}$ normal to both the plane of the loop and $\mathbf{n}$), and we let the segments parallel to $\mathbf{n}$ become infinitesimally short. We can write from equation (1.59) and Stokes' theorem (i.e., equation (1.36)) that

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = (\mathbf{t}\times\mathbf{n})\cdot(\mathbf{E}_2-\mathbf{E}_1)\Delta l = 0, \tag{1.77}$$

with Δl the length of the segments of C that are parallel to the boundary. Since $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$, and the orientation of the vector $\mathbf{t}$ (and, therefore, the orientation of the contour C) is arbitrary on the surface of the boundary, we can transform equation (1.77) to

$$\mathbf{n}\times(\mathbf{E}_2-\mathbf{E}_1) = 0. \tag{1.78}$$

That is, *the component of the electric field tangential to the boundary is continuous*.

1.6.1 Single-layer charge distributions

The problem of a single-layer surface charge was actually solved above where it was found that

$$\begin{aligned}(\mathbf{E}_2-\mathbf{E}_1)\cdot\mathbf{n} &= \frac{\sigma}{\varepsilon_0}\\ (\mathbf{E}_2-\mathbf{E}_1)\times\mathbf{n} &= 0.\end{aligned} \tag{1.79}$$

An equation for the scalar potential can also be derived for this problem by replacing ρd^3x with σda in equation (1.57). We then find the potential at any point in space from

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,da'. \tag{1.80}$$

Although the continuity or discontinuity of the components of the electric field are well specified from equations (1.79), it is not clear from equation (1.80) if the potential is continuous or not, or even if it is finite. For example, it is seen that the integrand becomes improper when the point of observation is located on the surface delimiting the boundary. One should, therefore, verify the continuity and finiteness of Φ, and we will make use of the following limiting process to do so.

We assume that the surface-charge density $\sigma(\mathbf{x}')$ is everywhere bounded, and we circumscribe a small disk or radius α centered on a point of observation $\mathbf{x}$ located on the boundary. First, it is clear that the potential due the surface charges outside the disk is a bounded and continuous function of position in the neighborhood of $\mathbf{x}$. If $\sigma_{\max}$ is the maximum value that the surface-charge density can take over the disk, it must be true that

$$\frac{1}{4\pi\varepsilon_0}\left|\int_{\text{disk}} \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,da'\right| < \frac{\sigma_{\max}}{2\varepsilon_0}\int_0^{\alpha}\frac{1}{r}\,r\,dr = \frac{\alpha\sigma_{\max}}{2\varepsilon_0}, \tag{1.81}$$

where $r = |\mathbf{x}'-\mathbf{x}|$. Since from equation (1.81) this contribution to the potential vanishes with the radius of the disk, we find that *the potential due to a surface-charge distribution is a bounded, continuous function of position at all points, both on and off the surface*. This result can easily be extended to volume-charge distributions by considering a sphere centered on the position of observation instead of a disk.

1.6.2 ***Double- or dipole-layer distributions***

A dipole-layer can be imagined as a surface generated by spreading positive charge of density $+\sigma(\mathbf{x})$ over the *positive* side of a regular surface S, and an identical distribution of opposite sign on its *negative* side. The result is a double-layer of charges separated by an infinitesimal distance $d(\mathbf{x})$ (see Figure 1.4). The dipole moment per unit area, of strength $D(\mathbf{x})$, is a vector directed along the unit normal vector $\mathbf{n}$ going from the negative to the positive sides of S, and is defined by the following limit

$$\mathbf{D}(\mathbf{x}) = \mathbf{n}\,D(\mathbf{x}) = \mathbf{n}\lim_{d(\mathbf{x})\to 0}\sigma(\mathbf{x})d(\mathbf{x}). \tag{1.82}$$

Using equation (1.80), we can express the potential due to the layer as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_{S_+}\frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,da' - \frac{1}{4\pi\varepsilon_0}\int_{S_-}\frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'+\mathbf{n}d|}\,da'. \tag{1.83}$$

Since the Taylor expansion of any function ψ is defined by

$$\psi(\mathbf{x}+\mathbf{a}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{a}\cdot\nabla)^n \psi(\mathbf{x}), \tag{1.84}$$

we can approximate

$$\frac{1}{|\mathbf{x}-\mathbf{x}'+\mathbf{n}d|} \simeq \frac{1}{|\mathbf{x}-\mathbf{x}'|} - d\,\mathbf{n}\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right), \tag{1.85}$$

where ∇' denotes the gradient relative to the components of $\mathbf{x}'$. Inserting equation (1.85) into equation (1.83) we find, in the limit $d(\mathbf{x}') \to 0$,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int_S \mathbf{D}(\mathbf{x}')\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) da'. \tag{1.86}$$

Since

$$\mathbf{n}\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) da' = -\frac{\cos(\theta)}{|\mathbf{x}-\mathbf{x}'|^2} da' = -d\Omega, \tag{1.87}$$

where $d\Omega$ is the solid angle subtended by da' at the point of observation (see Figure 1.5), then equation (1.86) can be written as

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0} \int_S D(\mathbf{x}')\, d\Omega. \tag{1.88}$$

We can use equation (1.88) to test the continuity (or lack thereof) of the potential across the dipole-layer. To do so, we use the same technique as was used for the single-layer surface-charge distribution, and separate the layer in a small disk centered on the point of observation and a remainder. Obviously, the contribution of the remainder is continuous across the plane of the hole occupied by the disk. If the disk is made infinitesimally small, then the dipole-layer can be considered as being constant and taken out of the integral. It is then seen that the potential just below the disk is given by

$$\Phi_- = -\frac{D}{4\pi\varepsilon_0} \int_{\text{disk}} d\Omega = -\frac{D}{2\varepsilon_0}, \tag{1.89}$$

since the disk covers half of space. Likewise, we find that the potential on the outer side of the disk is given by $\Phi_+ = D/2\varepsilon_0$, and that there is a discontinuity in the potential given by

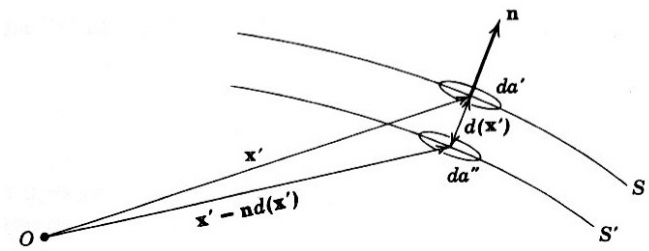

Figure 1.4 – Dipole-layer geometry.

$$\Phi_+ - \Phi_- = \frac{D}{\varepsilon_0}. \tag{1.90}$$

Using a pillbox, as we did earlier, that straddle the double-layer boundary, it is straightforward to see that the component of the electric field normal to the surface is now continuous since there is a net charge of zero within the box. That is,

$$\left(\mathbf{E}_+ - \mathbf{E}_-\right)\cdot\mathbf{n} = 0. \tag{1.91}$$

On the other hand, the component of the electric field tangential to the surface is not continuous. We now use the closed rectangular contour C of Figure 1.3, and a limiting process to determine the nature of the discontinuity. We denote by 1 the left corner located in medium 2, and proceed counterclockwise from there for 2, 3, and 4, and we use the notation Φ_i, with $i = 1, 2, 3, 4$, for the potential at each of the corners points. We know from equation (1.90) that the potential difference between points 1 and 2 is

$$\Phi_1 - \Phi_2 = \frac{D}{\varepsilon_0}, \tag{1.92}$$

corners points. We know from equation (1.90) that the potential difference between points 1 and 2 is

$$\Phi_1 - \Phi_2 = \frac{D}{\varepsilon_0}, \tag{1.93}$$

where we have taken the value of the dipole strength to be D at these points. With Δl the length of the segments parallel to the surface (boundary), the dipole strength at point 3 will be $D + \nabla D \cdot \Delta\mathbf{l}$. The potential difference between these two points will then be

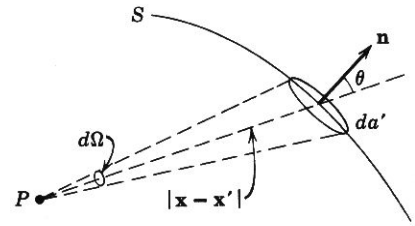

Figure 1.5 – The solid angle subtended by da' at the point of observation P.

$$\Phi_4 - \Phi_3 = \frac{1}{\varepsilon_0}\left(D + \nabla D \cdot \Delta\mathbf{l}\right). \tag{1.94}$$

Now, we must also have

$$\left(\Phi_2 - \Phi_1\right) + \left(\Phi_3 - \Phi_2\right) + \left(\Phi_4 - \Phi_3\right) + \left(\Phi_1 - \Phi_4\right) = 0, \tag{1.95}$$

or,

$$-\frac{D}{\varepsilon_0} + \left(\Phi_3 - \Phi_2\right) + \frac{1}{\varepsilon_0}\left(D + \nabla D \cdot \Delta\mathbf{l}\right) + \left(\Phi_1 - \Phi_4\right) = 0. \tag{1.96}$$

Abbreviating $\Delta\Phi_+ = \Phi_4 - \Phi_1$ and $\Delta\Phi_- = \Phi_3 - \Phi_2$, equation (1.96) reduces to

$$\frac{\Delta\Phi_+}{\Delta l} - \frac{\Delta\Phi_-}{\Delta l} = \frac{1}{\varepsilon_0}\nabla D \cdot \mathbf{h}, \tag{1.97}$$

where $\mathbf{h} = \Delta\mathbf{l}/\Delta l$ and is, therefore, a unit vector tangent to the boundary. If we take the limit when $\Delta l \to 0$, we have

$$\mathbf{E}_+ \cdot \mathbf{h} = -\lim_{\Delta l \to 0} \frac{\Delta\Phi_+}{\Delta l}, \tag{1.98}$$

and a similar relation between $\mathbf{E}_-$ and $\Delta\Phi_-$ (see equation (1.58)). Equation (1.97) then becomes

$$\left(\mathbf{E}_+ - \mathbf{E}_-\right) \cdot \mathbf{h} = -\frac{1}{\varepsilon_0}\nabla D \cdot \mathbf{h}. \tag{1.99}$$

Finally, since from the previous definitions of $\mathbf{n}$ and $\mathbf{t}$ we can also write $\mathbf{h} = \mathbf{n} \times \mathbf{t}$, and $\mathbf{a} \cdot \left(\mathbf{b} \times \mathbf{c}\right) = \mathbf{c} \cdot \left(\mathbf{a} \times \mathbf{b}\right)$, we can transform equation (1.99) to

$$\left(\mathbf{E}_{+}-\mathbf{E}_{-}\right)\times\mathbf{n}=-\frac{1}{\varepsilon_0}\nabla D\times\mathbf{n}. \tag{1.100}$$

In equation (1.100) we dropped the dot product with the unit vector $\mathbf{t}$ from both sides of the equality, since the orientation of the contour C (and therefore the orientation of $\mathbf{t}$) is arbitrary. It is seen from this result that the component of the electric field tangential to the boundary is discontinuous in general, unless the strength of the dipole-layer is constant along the surface (i.e. $\nabla D\times\mathbf{n}=0$).

1.7 Green's Theorem, the Poisson and Laplace Equations, and the Scalar Potential

Equation (1.57) for the definition of the scalar potential really only works well when the charges involved in the problem at hand are localized in space, and not outside of the region of interest. Unfortunately, this is not always the case. We are often required to take into account the presence of other charges or fields located in, or emanating from, remote regions by setting appropriate boundary conditions at the confines of a volume suitable to the problem at hand. This is done using equation (1.42) for Green's theorem, while setting

$$\phi=\Phi \qquad \text{and} \qquad \psi=\frac{1}{R}\equiv\frac{1}{\left|\mathbf{x}-\mathbf{x}'\right|}. \tag{1.101}$$

Now, let's consider a bounded charge density $\rho(\mathbf{x}')$ distributed somewhere in space, and a volume $V=V_1$ delimited by a surface S. It is not necessary that S encloses all of the charge, or even any of it. With $\mathbf{x}$ the observation point, the function ψ is improper at $\mathbf{x}=\mathbf{x}'$ (or $R=0$), we will therefore exclude any singularity by circumscribing $\mathbf{x}$ with a small sphere S_1 of radius r_1 (see Figure 1.6). The volume V is then bounded externally by S and internally by S_1. Within V both ϕ and ψ now satisfy the requirements of Green's theorem, which can now be written as (with $\nabla^2\psi=0$)

$$\int_V\frac{\nabla^2\Phi}{R}d^3x'=\int_{S+S_1}\left[\frac{1}{R}\frac{\partial\Phi}{\partial n'}-\Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]da'. \tag{1.102}$$

The left-hand side of equation (1.102) can be simplified by making use of the Poisson equation. We therefore insert equation (1.70) into equation (1.102) and we now have

$$-\frac{1}{\varepsilon_0}\int_V\frac{\rho(\mathbf{x}')}{R}d^3x'=\int_{S+S_1}\left[\frac{1}{R}\frac{\partial\Phi}{\partial n'}-\Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]da'. \tag{1.103}$$

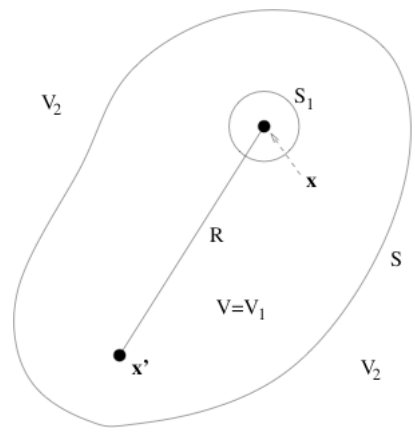

Figure 1.6 – Application of Green's theorem to a region V bounded externally by a surface S and internally by a sphere S_1.

We can also simplify the right-hand side of equation (1.103) by noting that over the sphere S_1, for which the unit normal vector $\mathbf{n}'$ is oriented radially inward, the following relations hold

$$\frac{\partial \Phi}{\partial n'} = -\frac{\partial \Phi}{\partial R}, \quad \text{and} \quad \left.\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right|_{R=r_1} = \frac{1}{r_1^2}. \tag{1.104}$$

Furthermore, if r_1 is chosen to be infinitesimally small, then Φ and $\partial\Phi/\partial R$ are basically constant over the sphere (which is centered at $\mathbf{x}$) and we have

$$\int_{S_1}\left[\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]da' = -\frac{1}{r_1}4\pi r_1^2\left.\frac{\partial \Phi}{\partial R}\right|_{\mathbf{x}} - \frac{1}{r_1^2}4\pi r_1^2\Phi(\mathbf{x}). \tag{1.105}$$

Taking the limit $r_1 \to 0$ we finally get

$$\boxed{\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')}{R}d^3x' + \frac{1}{4\pi}\int_S\left[\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]da'} \tag{1.106}$$

We know, from the discussion that follows equation (1.81), that the first integral on the right-hand side of equation (1.106) is valid even when the extent of the volume is such that $R = 0$ is possible at some point. *Equation (1.106) is thus seen to be the solution to the Poisson equation* $\nabla^2\Phi = -\rho/\varepsilon_0$. In the particular case where the volume V contains no charge, the equation for the potential reduces to

$$\Phi(\mathbf{x}) = \frac{1}{4\pi}\int_S\left[\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]da'. \tag{1.107}$$

It is, therefore, apparent that the surface integral appearing in equations (1.106) and (1.107) represents the contribution to the potential at $\mathbf{x}$ of all charges that are exterior to V (the volume integral in equation (1.106) being, obviously, the contribution on the charges inside of it). It then follows that *equation* (1.107) *is the solution to the Laplace equation* $\nabla^2\Phi = 0$. If there are no charges exterior to V, the surface integral must vanish and we recover equation (1.57) from equation (1.106) for the value of the potential $\Phi(\mathbf{x})$, as expected.

Finally, referring to equations (1.80) and (1.86), we note that the two terms arising from the surface integral in equation (1.106) are equivalent to the potential due to, respectively, a single-layer and a dipole-layer surface-charge distributions spread over S *when*

$$\sigma(\mathbf{x}') = \varepsilon_0 \frac{\partial \Phi}{\partial n'} \quad \text{and} \quad D(\mathbf{x}') = \varepsilon_0 \Phi(\mathbf{x}'). \tag{1.108}$$

Therefore, the charges outside S can (mathematically) be replaced by equivalent single- and double-layer distributions, of which the densities are given by equations (1.108), without modifying in any way the potential at the interior points. *It is important to note, however, that the potential due to these surface distributions does not correspond at all to that arising from the true charge distribution outside* S. *In fact, we know from the first of equations (1.79) that the component of the electric field normal to the surface is discontinuous at* S *such that*

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \mathbf{n}' = \frac{\sigma}{\varepsilon_0}, \tag{1.109}$$

or,

$$\left(\frac{\partial \Phi}{\partial n'}\right)\bigg|_1 - \left(\frac{\partial \Phi}{\partial n'}\right)\bigg|_2 = \frac{\sigma}{\varepsilon_0}, \tag{1.110}$$

where the indices 1 and 2 refer to the inner and outer sides of S, *respectively (see Figure 1.6). But since the normal derivative in equation (1.108) is calculated inside* S, *it follows from equation (1.110) that*

$$\left(\frac{\partial \Phi}{\partial n'}\right)\bigg|_2 = 0. \tag{1.111}$$

Moreover, from equation (1.93), the discontinuity in the potential at S *is given by*

$$\Phi_1 - \Phi_2 = \frac{D}{\varepsilon_0}, \tag{1.112}$$

and, substituting the second of equations (1.108) for Φ_1*, we find that*

$$\Phi_2 = 0. \tag{1.113}$$

Applying equation (1.106) to V_2*, with the fact that* $\rho = 0$ *everywhere in that region, tells us that the constraint* $\nabla^2\Phi = 0$ *must be fulfilled by the corresponding solution. Consequently, the potential within* V_2 *is completely defined by its boundary conditions at* S *(i.e.,* Φ_2 and $\left(\partial\Phi/\partial n'\right)\big|_2$*), but these have already been shown to vanish through equations (1.113) and (1.111). We conclude that* the equivalent surface layers specified by equations (1.108) are just those required to reduce both the potential and the electric field to zero at every point outside S. *As was mentioned before, this does not correspond to the actual potential generated by the true charges outside* S.

Example

Consider the problem of a charge q located at a distance Z away from a point at $\mathbf{x}$. We know Coulomb's Law that the potential at the observer is simply given by

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0 Z}. \tag{1.114}$$

However, if we circumscribe the point at $\mathbf{x}$ with an imaginary sphere of volume V enclosed in a surface S and devoid of any other charge, we should be able to obtain equation (1.114) using Green's theorem (i.e., equation (1.106)). We denote the radius of the sphere by R, and the vector linking the charge to a point on S by $\mathbf{r}$ (see Figure 1.7). We use spherical coordinates, such that a point on S has the coordinates (with $\mathbf{x}$ as origin)

$$\begin{aligned}\mathbf{R} &= x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z \\ &= R\sin(\theta)\cos(\varphi)\mathbf{e}_x + R\sin(\theta)\sin(\varphi)\mathbf{e}_y + R\cos(\theta)\mathbf{e}_z.\end{aligned} \tag{1.115}$$

Since there are no charges within the volume V, the potential at $\mathbf{x}$ will be given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi}\int_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] da'. \tag{1.116}$$

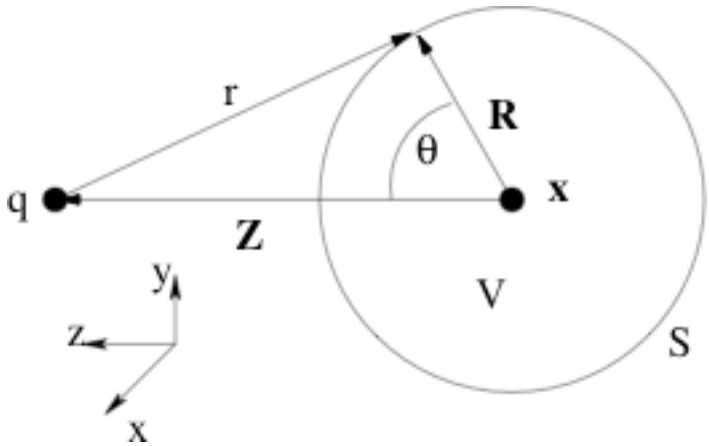

Figure 1.7 – The potential of a point charge q at $\mathbf{x}$.

Referring to Figure 1.7, we see that at the surface S

$$
\begin{aligned}
\mathbf{n}' &= \mathbf{e}_R \\
\frac{\partial}{\partial n'} &= \mathbf{n}' \cdot \nabla = \frac{\partial}{\partial R} \\
\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) &= -\frac{1}{R^2} \\
\Phi &= \frac{q}{4\pi\varepsilon_0 |\mathbf{r}|} = \frac{q}{4\pi\varepsilon_0 \left(R^2 - 2RZ\cos(\theta) + Z^2\right)^{1/2}} \\
\frac{\partial \Phi}{\partial n'} &= \frac{q}{4\pi\varepsilon_0} \frac{Z\cos(\theta) - R}{\left(R^2 - 2RZ\cos(\theta) + Z^2\right)^{3/2}}.
\end{aligned}
\tag{1.117}
$$

Using these equations, we get for the integrand of equation (1.116)

$$
\begin{aligned}
\frac{1}{R}\frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) &= \frac{q}{4\pi\varepsilon_0} \frac{Z}{R^2} \frac{Z - R\cos(\theta)}{\left(R^2 - 2RZ\cos(\theta) + Z^2\right)^{3/2}} \\
&= -\frac{q}{4\pi\varepsilon_0} \frac{Z}{R^2} \frac{\partial}{\partial Z}\left[\frac{1}{\left(R^2 - 2RZ\cos(\theta) + Z^2\right)^{1/2}}\right].
\end{aligned}
\tag{1.118}
$$

Inserting this equation into equation (1.116) we can calculate the potential at $\mathbf{x}$

$$
\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{q}{16\pi^2\varepsilon_0}\frac{Z}{R^2}\frac{\partial}{\partial Z}\left[\int_0^{2\pi} d\varphi \int_0^{\pi} \frac{R^2\sin(\theta)d\theta}{\left(R^2-2RZ\cos(\theta)+Z^2\right)^{1/2}}\right] \\
&= -\frac{qZ}{8\pi\varepsilon_0}\frac{\partial}{\partial Z}\left[\frac{1}{RZ}\sqrt{R^2-2RZ\cos(\theta)+Z^2}\Big|_0^{\pi}\right] \\
&= -\frac{qZ}{8\pi\varepsilon_0}\frac{\partial}{\partial Z}\left[\frac{1}{RZ}\left(\sqrt{R^2+2RZ+Z^2}-\sqrt{R^2-2RZ+Z^2}\right)\right] \\
&= -\frac{qZ}{8\pi\varepsilon_0}\frac{\partial}{\partial Z}\left[\frac{1}{RZ}\left(|R+Z|-|R-Z|\right)\right].
\end{aligned}
\tag{1.119}
$$

But since, from Figure 1.7, $Z > R$

$$
\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{qZ}{8\pi\varepsilon_0}\frac{\partial}{\partial Z}\left[\frac{1}{RZ}(R+Z-Z+R)\right] \\
&= -\frac{qZ}{8\pi\varepsilon_0}\frac{\partial}{\partial Z}\left[\frac{2}{Z}\right],
\end{aligned}
\tag{1.120}
$$

and finally

$$
\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0 Z}. \tag{1.121}
$$

We find, then, that the field calculated using the boundary conditions on the (arbitrary) surface S (within which there are no charges) gives the same result as that simply obtained with Coulomb's Law (i.e., equation (1.114)).

It is important to realize, as can be suspected from the previous example, that the boundary conditions (Φ and $\partial\Phi/\partial n'$ on the surface, for the so-called *Cauchy boundary conditions*) cannot be specified arbitrarily. In practice, one usually defines an electrostatic boundary-value problem by specifying either the potential (Φ for the *Dirichlet boundary conditions*), or the electric field (or $\partial\Phi/\partial n'$, for the *Neumann boundary conditions*) at the boundary. Independent choices for the two types of boundary conditions will generally lead to two different solutions for the potential within the surface. In other words, only when Φ and $\partial\Phi/\partial n'$ are consistently specified (as in the problem above) can the Cauchy boundary conditions be used.

1.8 *Solutions Using the Green Function*

Earlier, when using Green's theorem to find an expression for the scalar potential, we set

$$
\phi = \Phi \qquad \text{and} \qquad \psi = \frac{1}{R} \equiv \frac{1}{|\mathbf{x}-\mathbf{x}'|}. \tag{1.122}
$$

It turns out, however, that the solution thus obtained (i.e., equation (1.106)) is not the most general one can get. In fact, we can set instead of the last of equations (1.122)

$$\psi \equiv G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}'), \tag{1.123}$$

with

$$\nabla^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}'), \tag{1.124}$$

and, therefore, the function $F(\mathbf{x},\mathbf{x}')$ must satisfy

$$\nabla^2 F(\mathbf{x},\mathbf{x}') = 0 \tag{1.125}$$

within the volume V. With these conditions, the application of Green's theorem (using a procedure equivalent to that which led to equation (1.106) yields

$$\boxed{\begin{aligned}\Phi(\mathbf{x}) = &\frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' \\ &+\frac{1}{4\pi}\int_S\left[G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}\right]da'\end{aligned}} \tag{1.126}$$

The advantage of this formulation arises from the presence of $F(\mathbf{x},\mathbf{x}')$, as it provides us with the necessary degree of freedom to select the desired type of boundary conditions (i.e., either Dirichlet or Neumann). This is because the condition expressed by equation (1.125) equates the contribution of $F(\mathbf{x},\mathbf{x}')$ to the potential to that of a system of charges external to V. More precisely, we can choose the potential $F(\mathbf{x},\mathbf{x}')/4\pi\varepsilon_0$ such that, when combined with that of the point charge inside V and present in the Green function (i.e., $\left[4\pi\varepsilon_0|\mathbf{x}-\mathbf{x}'|\right]^{-1}$), the total potential $G(\mathbf{x},\mathbf{x}')/4\pi\varepsilon_0$ on, or its outward derivative to, the surface S has the required characteristics. For the **Dirichlet boundary conditions**, we specify that the total potential $G_D(\mathbf{x},\mathbf{x}')/4\pi\varepsilon_0$ cancels everywhere on the surface, or

$$G_D(\mathbf{x},\mathbf{x}') = 0, \qquad \text{for } \mathbf{x}' \text{ on } S, \tag{1.127}$$

and the potential within V is determined (from equation (1.126)), if it is known on S, by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\int_S \Phi\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'}da'. \tag{1.128}$$

On the other hand, for the **Neumann boundary conditions** we cannot set $\partial G_N/\partial n' = 0$ since from the divergence theorem

$$\begin{aligned}\int_V \nabla^2 G_N(\mathbf{x},\mathbf{x}')d^3x' &= \int_V \nabla\cdot\nabla G_N(\mathbf{x},\mathbf{x}')d^3x' \\ &= \int_S \mathbf{n}'\cdot\nabla G_N(\mathbf{x},\mathbf{x}')da' \\ &= \int_S \frac{\partial G_N(\mathbf{x},\mathbf{x}')}{\partial n'}da',\end{aligned} \tag{1.129}$$

and from equation (1.124) $\int_V \nabla^2 G_N(\mathbf{x},\mathbf{x}')d^3x' = -4\pi$. The simplest boundary condition for the outward derivative of the Green function is therefore

$$\frac{\partial G_N(\mathbf{x},\mathbf{x}')}{\partial n'} = -\frac{4\pi}{S}, \qquad \text{for } \mathbf{x}' \text{ on } S. \tag{1.130}$$

The potential within V is determined if its outward derivative is known on S

$$\Phi(\mathbf{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G_N(\mathbf{x},\mathbf{x}')d^3x' + \frac{1}{4\pi}\int_S \frac{\partial\Phi}{\partial n'}G_N(\mathbf{x},\mathbf{x}')da', \tag{1.131}$$

where $\langle\Phi\rangle_S$ is the average of the potential over S.

It is in principle possible to solve any electrostatic boundary-value problem using the Green function formalism. This statement implies, however, that the Green function is known a priori, or that it can be evaluated. Unfortunately, this is not always true as the form of the Green function can only be evaluated for relatively simple surface geometries. For this reason, one often resorts to other techniques that are not necessarily closely related to this formalism.